A Method of Mechanically Compensating the Rotation of the Field of a Siderostat. By H. C. Plummer, M.A.

1. The compensation of the rotation of the field of a siderostat, which is necessary for photographic purposes, can be effected doubtless in a great variety of ways. Three methods of achieving this object have been suggested by Professor Turner in the *Monthly Notices* for 1901 January. The third method is particularly elegant; but it is possible, I think, to go still further in the direction of mechanical simplicity. That this is the case I hope the device to be described in this note will make clear.

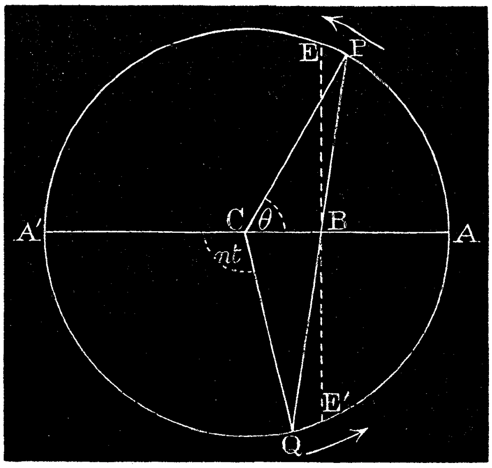

FIG. 1.

2. Let C be the centre of the circle $x^2+y^2=a^2$, and AA′ a diameter. If ACP $=\theta$ and ACQ $=\phi$ (fig. 1), both angles being measured in the same direction, the equation of PQ is

$$x \cos \tfrac{1}{2}(\theta+\phi)+y \sin \tfrac{1}{2}(\theta+\phi)=a \cos \tfrac{1}{2}(\theta-\phi).$$

Hence if PQ cuts AA′ in B, and CB $=b$

$$\frac{b}{a}=\frac{\cos \frac{1}{2}(\theta-\phi)}{\cos \frac{1}{2}(\theta+\phi)}=\frac{1+\tan \frac{1}{2}\theta \tan \frac{1}{2}\phi}{1-\tan \frac{1}{2}\theta \tan \frac{1}{2}\phi}$$

$$\therefore \tan \tfrac{1}{2}\theta \tan \tfrac{1}{2}\phi=-\frac{a-b}{a+b}$$

Let now ACQ $=\phi=\pi+nt$.

$$\therefore \tan \tfrac{1}{2}\theta=\frac{a-b}{a+b}\tan \tfrac{1}{2}nt=\mathrm{K}\tan \tfrac{1}{2}nt$$

if $\dfrac{b}{a}=\dfrac{1-\mathrm{K}}{1+\mathrm{K}}$. But the rotation which is to be compensated is given by the equation

$$\tan \tfrac{1}{2}\theta=\mathrm{K}\tan \tfrac{1}{2}nt$$

where $$K = \frac{\cos \frac{1}{2}(\rho+\delta)}{\cos \frac{1}{2}(\rho-\delta)} = \frac{1 - \tan \frac{1}{2}\rho \tan \frac{1}{2}\delta}{1 + \tan \frac{1}{2}\rho \tan \frac{1}{2}\delta}$$

or $$\frac{1-K}{1+K} = \tan \tfrac{1}{2}\rho \tan \tfrac{1}{2}\delta$$

Hence it is evident that if $b = a \tan \frac{1}{2}\rho \tan \frac{1}{2}\delta$, and if Q describes the circle uniformly in one day, P also describes the circle in one day, not uniformly, but with exactly the same motion as that of the field of a siderostat set for a star whose N.P.D. is δ. The angle ρ is the angle between the polar axis and the ray reflected in a fixed direction. If the siderostat mirror is placed south of the telescope, the rotation of the field will be in the same direction for all stars which are visible, and the limits of δ are 0 and $\pi - \rho$, and of $\tan \frac{1}{2}\rho \tan \frac{1}{2}\delta$ 0 and 1. That is to say, the limits of b are 0 and a, or B may have any position along the radius CA. The diameter AA′ is in the meridian of the instrument. When the star at the centre of the field is at upper culmination on the instrumental meridian PQ coincides with AA′, and is rotating with minimum velocity. When the same star comes to lower culmination PQ coincides with A′A, and is rotating with maximum velocity.

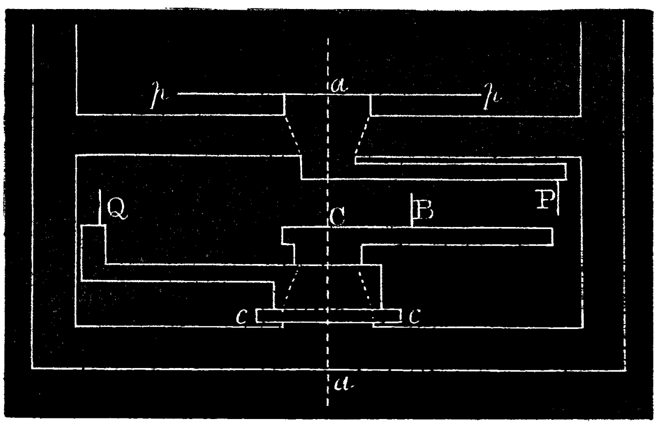

Fig. 2.

3. It is easy to imagine a mechanical arrangement which will satisfy these geometrical conditions. The points P and Q may be represented by two pins mounted on two arms of equal length which are capable of rotating about the same axis. A third bar perpendicular to this axis must be fixed in the meridian of the instrument, and on this we must have the means of accurately adjusting a third pin at a given distance from the axis. This pin serves to fix the position of the point B. A straight slotted bar is pivoted on the pin Q, and passing over the pins B and P keeps the three collinear. Then if the bar carrying Q rotates by clockwork with uniform diurnal motion, that carrying P rotates

in the required manner, and communicates the appropriate motion to the plate-carrier *p a p* (fig. 2), to which it is attached. It is outside my province to discuss here the mechanical details by which the device suggested may best be practically realised. The section in the meridian plane is only given in the hope that it may make a little clearer the practicability of the general method. In this figure *c c* represents the circle to which uniform motion is communicated, but the slotted bar QBP has been omitted.

4. Assuming that a practical design has been found, it is important to study the effect of inaccuracies of adjustment. In setting the pin B we may make an error δb in $b = a \tan \frac{1}{2}\rho \tan \frac{1}{2}\delta$. In the second place the motion may not be correct in phase, that is, the compensating bar PQ may cross the instrumental meridian a small interval of time δt in advance of the star at the centre of the field. When, therefore, the field has turned through an angle θ such that $\tan \frac{1}{2}\theta = \mathrm{K} \tan \frac{1}{2} nt$ the plate has turned through an angle $\theta + \delta\theta$ such that

$$\tan \tfrac{1}{2}(\theta + \delta\theta) = (\mathrm{K} + \delta\mathrm{K}) \tan \tfrac{1}{2} n(t + \delta t)$$

$$\therefore\ \delta\theta = 2 \cos^2 \tfrac{1}{2}\theta \{\delta\mathrm{K} \,.\, \tan \tfrac{1}{2} nt + \tfrac{1}{2}\mathrm{K}n \sec^2 \tfrac{1}{2} nt \,.\, \delta t\}$$

$$= \sin\theta \,.\, \frac{\delta\mathrm{K}}{\mathrm{K}} + \frac{n}{\mathrm{K}} \,.\, \frac{\mathrm{K}^2 + \tan^2\frac{1}{2}\theta}{1 + \tan^2\frac{1}{2}\theta} \,.\, \delta t$$

$$= \sin\delta \,.\, \frac{\delta\mathrm{K}}{\mathrm{K}} + \frac{n}{2\mathrm{K}}\{1 + \mathrm{K}^2 - (1 - \mathrm{K}^2)\cos\theta\} \,.\, \delta t.$$

Now
$$\frac{\delta\mathrm{K}}{\mathrm{K}} = -\frac{\delta b}{a - b} - \frac{\delta b}{a + b} = -\frac{2a}{a^2 - b^2} \,.\, \delta b$$

$$\tfrac{1}{2}\left(\frac{1}{\mathrm{K}} + \mathrm{K}\right) = \frac{a^2 + b^2}{a^2 - b^2}$$

$$\tfrac{1}{2}\left(\frac{1}{\mathrm{K}} - \mathrm{K}\right) = \frac{2ab}{a^2 - b^2}$$

Hence
$$\delta\theta = -\frac{2a \sin\theta}{a^2 - b^2} \,.\, \delta b + \frac{n}{a^2 - b^2}(a^2 + b^2 - 2ab\cos\theta)\delta t.$$

5. The actual value of $\delta\theta$ is of little or no consequence, and only concerns the orientation of the picture on the plate. What is of importance is the rate of change of $\delta\theta$. This is

$$\frac{d}{dt}(\delta\theta) = -\frac{2a}{a^2 - b^2}(\cos\theta \,.\, \delta b - b \sin\theta \,.\, n\delta t)\frac{d\theta}{dt}$$

$$= -\frac{2an}{(a^2 - b^2)^2}(a^2 + b^2 - 2ab\cos\theta)(\cos\theta \,.\, \delta b - b\sin\theta \,.\, n\delta t).$$

The critical values of this rate are given by

$$-(a^2 + b^2 - 2ab\cos\theta)(\sin\theta \,.\, \delta b + b\cos\theta \,.\, n\delta t)$$
$$+ 2ab\sin\theta(\cos\theta \,.\, \delta b - b\sin\theta \,.\, n\delta t) = 0$$

or $$-(a^2+b^2)(\sin\theta\,.\,\delta b+b\cos\theta\,.\,n\delta t)$$
$$+2ab(\sin 2\theta\,.\,\delta b+b\cos 2\theta\,.\,n\delta t)=0$$

i.e. $$(a^2+b^2)\sin(\theta+a)=2ab\sin(2\theta+a)$$

if $$\tan a=b\,.\,n\delta t/\delta b$$

If the motion of the arm which carries the pin Q is obtained by gearing it to the hour circle of the siderostat the adjustment of the phase will be automatic. This would be a most convenient arrangement, and ought to minimise the error to be feared from this source. When $\delta t=0$ the critical values of the rate found above will occur at the positions

$$\theta=0,\ \pi,\ \pm\cos^{-1}\frac{a^2+b^2}{4ab}.$$

In one complete rotation the two errors of adjustment will cause a simple oscillation of the field with respect to the plate, of which the amplitude is

$$\frac{4a}{a^2-b^2}\left\{[\delta b]^2+b^2[n\delta t]^2\right\}^{\frac{1}{2}}$$

$$=\frac{2\sin\rho\sin\delta}{\cos\rho+\cos\delta}\left\{\left[\frac{\delta b}{b}\right]^2+[n\delta t]^2\right\}^{\frac{1}{2}}$$

$$=\frac{2\sin\rho}{\cos\rho+\cos\delta}\left\{[\Delta\delta]^2+\sin^2\delta[n\delta t]^2\right\}^{\frac{1}{2}}$$

where $\Delta\delta$ is the error in δ corresponding to the error δb in b. For since $b=a\tan\frac{1}{2}\rho\tan\frac{1}{2}\delta$

$$\delta b/b=\Delta\delta/\sin\delta$$

The fixed bar which carries the pin B will naturally be graduated so as to give the reading of δ directly.

6. When $\delta+\rho=\pi$, $K=0$, and $b=a$; that is to say, in the limit B coincides with A. If the telescope is due north of the siderostat, $\delta+\rho=\pi$ only for a star which culminates on the southern horizon. In the case of a cœlostat the condition is permanent. The representation of the motion of the field given in § 2 illustrates very well Professor Turner's remark on this case. For let EE′ (fig. 1) be the chord perpendicular to the diameter AA′, then P describes the arc E′AE while Q describes the arc EA′E′, and the arc EA′E′ while Q describes E′AE. Now in the limit the arc E′AE vanishes when B coincides with A, and the arc EA′E′ approaches the whole circumference. Hence we see that P coincides with A for twenty-four hours, and then describes the circle with infinite velocity.

7. This interesting example of a change of sign in passing through an infinite value may be considered in another way. We have

$$\frac{d\theta}{dt}=\frac{n}{a^2-b^2}(a^2+b^2-2ab\cos\theta)$$

$$=\frac{n}{2K}\{1+K^2-(1-K^2)\cos\theta\}$$

Hence the hodograph (turned through 90°) of a point rotating with the field may be represented on a scale proportional to K by the curve

$$r = 1 + K^2 - (1 - K^2)\cos\theta$$

which is, of course, a limaçon. Since $1 - K^2 < 1 + K^2$, the curve has no node. The vector in the direction $\theta = \pi$ is constant for all values of K. When $K = 1$, the curve is a circle concentric with the pole. When $K = 0$ the hodograph is a cardioide with a cusp at the pole. The parameter is finite, and this evidently corresponds to an infinite angular velocity, because the hodograph has been drawn on a scale proportional to K. In this way we see, not only that the velocity is infinite, but that it is proportional at every point to the radius vector of a cardioide drawn in the same direction.

8. It may not be out of place to add some indication of the way in which this simple device suggested itself. Consider two points describing circles whose radii are r and s, the first with the motion characteristic of the siderostat field, the second uniformly in the same period. Let the motions be projected on a diameter of each circle and let x and y be the projections of the moving radii, so that

$$x = r\cos\theta,\ y = s\cos nt$$

$$\therefore\ \tan^2\frac{1}{2}\theta = \frac{r-x}{r+x},\ \tan^2\frac{1}{2}nt = \frac{s-y}{s+y}$$

$$\therefore\ (r-x)(s+y) = K^2(s-y)(r+x)$$

$$\therefore\ xy(1-K^2) + sx(1+K^2) - ry(1+K^2) - rs(1-K^2) = 0$$

Hence, if the circles are concentric and the diameters of projection coincident, x and y determine two ranges which are related homographically. If further $s = r$ and the sign of y is changed, the relation becomes

$$xy(1-K^2) - rx(1+K^2) - ry(1+K^2) + r^2(1-K^2) = 0$$

or

$$\left(\frac{x}{r} - \frac{1+K^2}{1-K^2}\right)\left(\frac{y}{r} - \frac{1+K^2}{1-K^2}\right) = \frac{4K^2}{(1-K^2)^2}$$

and the ranges are in involution. The centre of the involution is at a distance $r(1+K^2)/(1-K^2)$ from the centre of the circle and the extremities of the diameter of projection are conjugate points. Hence the construction (fig. 3): Let PMP′, QNQ′ be chords of a circle which are constrained to remain perpendicular to the diameter AA′. The centre of the circle is C, and $CO = r(1+K^2)/(1-K^2)$. Let OMNHK be a Peaucellier linkage, such that AA′ are possible positions of MN. Then if Q describes the circle uniformly in twenty-four hours in the sense

indicated, P will move in the manner required to compensate the rotation of the siderostat field.

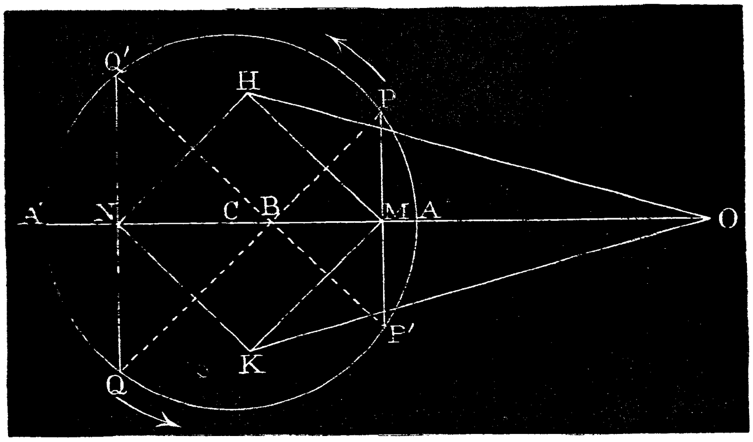

Fig. 3.

9. So far we have derived the motion of P from the rectilinear motions of M and N. The motion of N being simple harmonic, that of M may perhaps be called simple involutional. The method is not a practical one because for fields in the neighbourhood of the pole O is at a very great distance from C. But this method is directly related to the vastly simpler one already given. The parallel chords PP′ and QQ′ are conjugate lines of an involution pencil of which the vertex is at infinity, and the tangents at AA′ are also a conjugate pair. Let PQ, P′Q′ intersect AA′ in the point B. By Desargue's theorem every line intersects the circle and opposite sides of the inscribed quadrangle PP′ QQ′ in an involution range. Hence B is a double point of the involution of which AA′, MN are pairs, and it follows that B is a fixed point for every pair of the involution pencil. Its position is given by the equation

$$x^2(1-K^2)-2rx(1+K^2)+r^2(1-K^2)=0$$

of which the roots are $x=\frac{1-K}{1+K}$ and $x=\frac{1+K}{1-K}$. The former lies within the circle, and is the position determined independently in § 2. The latter is the double point outside, and does not give so convenient a solution of the practical problem. The points PQ, P′Q′, . . . are pairs of an involution range in the circle. This proposition, of which a proof has been given here, can be stated in a more general form. An involution pencil, of which the tangents from the vertex to a conic form a pair, determines an involution range on the conic.

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Further Investigation of the "Two Method" Personal Equation. By Walter W. Bryant.

In a former paper (*Monthly Notices*, 1898 March) I gave an analysis of the results of five years' determinations of the differences of personality between the eye-and-ear and galvanic method for H., A. C., and B., whenever a clock-star was observed by either of them both ways the same night, as part of the routine of the observatory at Greenwich.

The additional evidence since accumulated may throw fresh light upon some of the points then raised.

It may be well to note briefly, as before, that the quantity under discussion is very fairly represented by $t_h + t_c$ for H., who observes galvanically by the "sensorial" method, and by t_h for A. C. and B., who adopt the "muscular" method, t_h being the "reaction to sound," and t_c the time occupied in making a contact, which is theoretically eliminated in the "muscular" method.

Some of the more ordinary causes of variation may be classified as follows :—

A. Personal, the observer's physical, and especially nervous, condition with regard to—

(1) General health, age, &c.

(2) Time of year, time of day, temperature and other external influences.

(3) Comfort at the moment, depending upon observing position (standing, sitting, or reclining), including the inclination of the head with reference to the clock, &c., &c.

B. Impersonal, depending on instruments.

(1) The pricker chronograph is open to the objection that the slight tendency to stop the barrel may introduce a variation in the accuracy of the clock comparison depending upon the fraction of a second between the beats from the two clocks.

(2) Galvanic circuits are liable to vary in resistance, especially at contacts, with heat, cold, damp, &c. ; but this cause is probably insignificant.

(3) From a cause possibly connected with (1) the seconds of Clock Hardy, the transit clock, do not appear to be of the same length, the odd seconds giving a reading differing from that of the even ones.

(4) The force necessary to make a contact varies with the strength of the spring. A new spring tends to make all galvanic observations a little late. If this is delicately adjusted to avoid the difficulty it is found that some